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# Scattering for the radial 3D cubic focusing INLS equation

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In this work (recently published in J. Differential Equations [5]) we study the initial value problem (IVP) or also called the Cauchy problem for the cubic inhomogenous nonlinear Schrödinger equation (INLS) in $3D$

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^2 u = 0, & t \in \mathbb{R},\ x \in \mathbb{R}^3, \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where $u = u(t,x)$ is a complex-valued function in space-time $\mathbb{R} \times \mathbb{R}^3$ and $0 < b < 1/2$.

The scale-invariant Sobolev norm is $H^{s_c}(\mathbb{R}^N)$ with $s_c = \frac{b+1}{2}$ (Critical Sobolev index). We briefly review recent developments on the well-posedness theory for the general INLS equation

$$\begin{cases} i\partial_t u + \Delta u + |x|^{-b}|u|^\alpha u = 0, & x \in \mathbb{R}^N, \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

Genoud and Stuart [3], using the abstract theory developed by Cazenave [1] and some sharp Gagliardo-Nirenberg inequalities, showed that (2) is well-posed in $H^1(\mathbb{R}^N)$

- locally if $0 < \alpha < 2^*$,
- globally for small initial condition if $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$,
- globally for any initial condition if $0 < \alpha < \frac{4-2b}{N}$,
- globally if $\alpha = \frac{4-2b}{N}$, assuming $\|u_0\|_{L^2} < \|Q\|_{L^2}$,

where $Q$ is the ground state of the equation $-Q + \Delta Q + |x|^{-b}|Q|^{\frac{4-2b}{N}}Q = 0$ and $2^* = \frac{4-2b}{N-2}$, if $N \geq 3$ or $2^* = \infty$, if $N = 1, 2$. Recently, the second author in [4], using the contraction mapping principle based on the Strichartz estimates, proved that the IVP (2) is locally well-posed in $H^1(\mathbb{R}^N)$, for $0 < \alpha < 2^*$. Moreover, for $N \geq 2$, $\frac{4-2b}{N} < \alpha < 2^*$ these solutions are global in $H^1(\mathbb{R}^N)$ for small initial data. It is worth mentioning that Genoud and Stuart [3] consider $0 < b < \min\{2, N\}$, and the second author in [4] assume $0 < b < \widetilde{2}$, where $\widetilde{2} = N/3$ if $N = 1, 2, 3$ and $\widetilde{2} = 2$ if $N \geq 4$. This new restriction on $b$ is needed to estimate the nonlinear part of the equation in order to use the well known Strichartz estimates associated to the linear flow.

On the other hand, since

$$\|u_\delta\|_{L^2_x} = \delta^{-s_c}\|u\|_{L^2_x}, \quad \|\nabla u_\delta\|_{L^2_x} = \delta^{1-s_c}\|\nabla u\|_{L^2_x} \tag{3}$$

and

$$\left\||x|^{-b}|u_\delta|^4\right\|_{L^1_x} = \delta^{2(1-s_c)}\left\||x|^{-b}|u|^4\right\|_{L^1_x},$$

the following quantities enjoy a scaling invariant property

$$E[u_\delta]^{s_c}M[u_\delta]^{1-s_c} = E[u]^{s_c}M[u]^{1-s_c}, \;\; \|\nabla u_\delta\|^{s_c}_{L^2_x}\|u_\delta\|^{1-s_c}_{L^2_x} = \|\nabla u\|^{s_c}_{L^2_x}\|u\|^{1-s_c}_{L^2_x}, \tag{4}$$

where $E[u(t)] = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla u(t,x)|^2dx - \frac{1}{4}\left\||x|^{-b}|u|^4\right\|_{L^1_x}$ and $M[u(t)] = \int_{\mathbb{R}^3}|u(t,x)|^2dx$, which are calling Energy and Mass, respectively. These quantities were introduced in Holmer-Roudenko [6] in the context of mass-supercritical and energy-subcritical nonlinear Schrödinger equation (NLS), which is equation (1) with $b = 0$, and they were used to understand the dichotomy between blowup/global regularity. Indeed, in [6], the authors consider the $3D$ cubic NLS and proved that if the initial data $u_0 \in H^1(\mathbb{R}^3)$ is radial and satisfies

$$E(u_0)M(u_0) < E(Q)M(Q) \quad \text{and} \quad \|\nabla u_0\|_{L^2}\|u_0\|_{L^2} < \|\nabla Q\|_{L^2}\|Q\|_{L^2},$$

then the corresponding solution $u(t)$ of the Cauchy problem (1) (with $b = 0$) is globally defined and scatters[1] in $H^1(\mathbb{R}^3)$ where $Q$ is the ground state solution of the nonlinear elliptic equation $-Q+\Delta Q+|Q|^2Q = 0$. The subsequent work Duyckaerts-Holmer-Roudenko [2] has removed the radial assumption on the initial data. In both papers, they used the method of the concentration-compactness and rigidity technique employed by Kenig-Merle [7] in their study of the energy critical NLS. Inspired by these works, we investigate same problem for the IVP (1).

# 1 Main Result

Let $u_0 \in H^1(\mathbb{R}^3)$ be radial, and let $u$ the corresponding solution to (1) in $H^1$. Suppose

$$M[u]^{s_c}E[u]^{1-s_c} < M[Q]^{s_c}E[Q]^{1-s_c},$$

and if $\|u_0\|^{s_c}\|\nabla u_0\|^{1-s_c} < \|Q\|^{s_c}\|\nabla Q\|^{1-s_c}$, then $u$ is globally defined and scatters in $H^1$. That is, there exists $v \in H^1(\mathbb{R}^3)$ such that

$$\lim_{t\to\infty}\|u(t) - U(t)v\|_{H^1} = 0,$$

where $Q$ is the ground state of $-Q + \Delta Q + |x|^{-b}|Q|^2Q = 0$.

*The above result extends the work obtained by Holmer-Roudenko [6] to the INLS model. On the other hand, since the solutions of the INLS equation do not enjoy conservation of momentum, we were not able to use the same ideas introduced by Duyckaerts-Holmer-Roudenko [2] to remove the radial assumption.*

# Referências

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[1]Notice that, in this case the critical Sobolev index is $s_c = 1/2$.

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