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Existence result for an equation with (p-q)-laplacian and vanishing potentials

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Abstract

The main purpose of this wokd is to establish the existence of positive solution to a class of quasilinear elliptic equations involving the (p-q)-Laplacian operator. We consider a nonlinearity that can be subcritical at infinity and supercritical at the origin; we also consider potential functions that can vanish at infinity. The approach is based on variational arguments dealing with the mountain-pass lemma and an adaptation of the penalization method. In order to overcome the lack of compactness we modify the original problem and the associated energy functional. Finally, to show that the solution of the modified problem is also a solution of the original problem we use an estimate obtained by the Moser iteration scheme.

1 Introduction

In this work we consider a class of quasilinear elliptic equations involving the (p-q)-Laplacian operator of the form

$$\begin{cases} -\Delta_p u - \Delta_q u + a(x)\,|u|^{p-2}\,u + b(x)\,|u|^{q-2}\,u = f(u), & x \in \mathbb{R}^N; \\ \;u(x) > 0, \quad u \in D^{1,p}(\mathbb{R}^N) \cap D^{1,q}(\mathbb{R}^N), & x \in \mathbb{R}^N. \end{cases} \tag{1.1}$$

The m-laplacian operator is defined by $\Delta_m u(x) \equiv \operatorname{div}(|\nabla u(x)|^{m-2}\nabla u(x))$, for $m \in \{p,q\}$, where $2 \leqslant q \leqslant p < N$; the Sobolev space is defined by $D^{1,m}(\mathbb{R}^N) \equiv \{u \in L^{m^*}(\mathbb{R}^N) \colon (\partial u/\partial x_i)(x) \in L^m(\mathbb{R}^N), \quad 1 \leqslant i \leqslant N\}$, and the critical Sobolev exponent is given by $m^* \equiv Nm/(N-m)$, also for $m \in \{p,q\}$.

The nonlinearity $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous, nonnegative function that is not a pure power and can be subcritical at infinity and supercritical at the origin. More precisely, the following set of hypotheses on the nonlinearity f is used.

1. $\limsup_{s\to 0^+} sf(s)/s^{p^*} < +\infty$.
2. There exists $\tau \in (p,p^*)$ such that $\limsup_{s\to+\infty} sf(s)/s^{\tau} = 0$.
3. There exists $\theta > p$ such that $0 \leqslant \theta F(s) \leqslant sf(s)$ for every $s \in \mathbb{R}^+$, where we use the notation $F(s) \equiv \int_0^s f(t)\,\mathrm{d}t$.
4. $f(t) = 0$ for every $t \leqslant 0$.

Is is worth noticing that hypothesis (3) extends a well known condition which was first formulated by Ambrosetti and Rabinowitz. It states a sufficient condition to ensure that the energy funcional, associated in a natural way to this type of problem, verifies the Palais-Smale condition. Recall that a functional $J\colon D^{1,m}(\mathbb{R}^N) \to \mathbb{R}$ is said to verify the Palais-Smale condition at the level c if any sequence $(u_n)_{n\in\mathbb{N}} \subset D^{1,m}(\mathbb{R}^N)$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$, as $n \to +\infty$, possess a convergent subsequence. Hypothesis (3) also allows us to study the asymptotic behavior of the solution to the problem.

We also assume that the functions $a,b\colon \mathbb{R}^N \to \mathbb{R}$ are continuous and nonnegative. Moreover, the following set of hypotheses on the potential functions a and b is used.

1. $a \in L^{N/p}(\mathbb{R}^N)$ and $b \in L^{N/q}(\mathbb{R}^N)$.
2. $a(x) \leqslant a_\infty$ and $b(x) \leqslant b_\infty$ for every $x \in B_1(0)$, where $a_\infty, b_\infty \in \mathbb{R}^+$ are positive constants and $B_1(0)$ denotes the unitary ball centered at the origin.
3. There exist constants $\Lambda \in \mathbb{R}^+$ and $R_0 > 1$ such that $\dfrac{1}{R_0^{p^2/(p-1)}} \inf_{|x|\geqslant R_0} |x|^{p^2/(p-1)} a(x) \geqslant \Lambda$.

The interest in the study of this type of problem is twofold. On the one hand we have the physical motivations, since the quasilinear operator (p-q)- Laplacian has been used to model steady-state solutions of reaction-diffusion problems arising in biophysics, in plasma physics and in the study of chemical reactions. On the other hand we have the purely mathematical interest in these type of problems, mainly regarding the existence of positive solutions as well as multiplicity results. The assumptions $\liminf_{|x|\to+\infty} a(x) > 0$ and $\liminf_{|x|\to+\infty} b(x) > 0$ are used in many papers. In problem (1.1) we consider the exponents $2 \leqslant q \leqslant p < N$ and we allow the particular conditions $\liminf_{|x|\to+\infty} a(x) = 0$ and $\liminf_{|x|\to+\infty} b(x) = 0$, called the zero mass cases. These constitute the main features of our work.

2 Main result

Inspired mainly by Wu and Yang [4] regarding the (p-q)-Laplacian type operator, and by Alves and Souto [1] with respect to the set of hypotheses, our result reads as follows.

Theorem 2.1. *Consider $2 \leqslant q \leqslant p < N$ and suppose that the potential functions a and b verify the hypotheses (1), (2) and (3) and that the nonlinearity f verifies the hypotheses (1), (2), (3), and (4). Then there exists a constant $\Lambda^* = \Lambda^*(a_\infty, b_\infty, \theta, \tau, c_0)$ such that problem (1.1) has a positive solution for every $\Lambda \geqslant \Lambda^*$.*

We adapt the penalization method developed by del Pino and Felmer [3] to show our existence result. The basic idea can be described in the following way: first we modify the original problem and study its corresponding energy functional, showing that it verifies the geometry of the mountain-pass lemma and that every Palais-Smale sequence is bounded in an appropriate Sobolev space. Using the standard theory this implies that the modified problem has a solution; then we show, using the Moser iteration scheme, that the solution of the auxiliary problem verifies an estimate involving the $L^\infty(\mathbb{R}^N)$ norm; finally we use this estimate to show that the solution of the modified problem is also a solution of the original problem (1.1).

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